

# AN IMPROVED STRICHARTZ ESTIMATE FOR SYSTEMS WITH DIVERGENCE FREE DATA

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**ABSTRACT.** Using the div-curl inequalities of Bourgain-Brezis [1] and van Schaftingen [9], we prove an improved Strichartz estimate for systems of inhomogeneous wave and Schrodinger equations, for which the inhomogeneity is a divergence-free vector field at each given time. The novelty of the result is that one can allow  $L_x^1$  norms of the inhomogeneity in the right hand side of the estimate.

In this paper we are interested in improved Strichartz estimates for systems of inhomogeneous wave and Schrodinger equations, when the inhomogeneity is a divergence free vector field at any given time. The starting point is the following simple observation:

**Proposition 1.** *Suppose  $u: \mathbb{R}^{1+2} \rightarrow \mathbb{R}^2$  is a (weak) solution of the following system of wave equations*

$$\begin{cases} \square u = f \\ u|_{t=0} = u_0 \\ \partial_t u|_{t=0} = u_1 \end{cases}$$

where  $f = (f_1, f_2): \mathbb{R}^{1+2} \rightarrow \mathbb{R}^2$  is a divergence free vector field at each given time  $t$ , i.e.

$$\partial_{x_1} f_1 + \partial_{x_2} f_2 = 0$$

for each  $t$ . Then

$$\|u\|_{C_t^0 L_x^2} + \|\partial_t u\|_{C_t^0 \dot{H}_x^{-1}} \leq C \left( \|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-1}} + \|f\|_{L_t^1 L_x^1} \right).$$

Here  $\square = -\partial_t^2 + \Delta$  is the d'Alembertian acting componentwise on  $u$ , and  $\dot{H}^s$  is the homogeneous Sobolev space  $\dot{W}^{s,2}$ .

A remarkable feature in our estimate is that on the right hand side we only need the  $L_x^1$  norm of  $f$ , which is usually not possible in the classical energy (or Strichartz) inequalities. Our estimate is only possible because we have the additional structural assumption that  $f$  is a divergence free vector field at each time  $t$ . In fact if one tries to prove the Proposition using Sobolev embedding naively without using this divergence free assumption, say when

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$u_0 = u_1 = 0$ , then one would estimate, at any time  $t$ ,

$$\begin{aligned} \|u\|_{L_x^2} &= \left\| \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} f(s, x) ds \right\|_{L_x^2} \\ &\leq \int_0^t \left\| \frac{1}{\sqrt{-\Delta}} f(s, x) \right\|_{L_x^2} ds \\ &\leq \int_0^t \|R_1 f(s, x)\|_{L_x^1} + \|R_2 f(s, x)\|_{L_x^1} ds \end{aligned}$$

where  $R_j$  are the Riesz transforms on  $\mathbb{R}^2$ , which are unfortunately not bounded on  $L^1$ .

Before we state our more general results, we first give a short proof of Proposition 1. The proof relies on the following simple observation that we first learned from Bourgain-Brezis [1]:

**Lemma 1** (Bourgain-Brezis). *For each divergence free vector field  $F = (F_1, F_2)$  on  $\mathbb{R}^2$  with  $F \in L^1$ , there exists  $G \in L^2$  such that  $F_1 = \partial_{x_2} G$  and  $F_2 = -\partial_{x_1} G$  with  $\|G\|_{L^2} \leq C\|F\|_{L^1}$ .*

*Proof.* The assumption that  $\operatorname{div} F = 0$  allows one to find  $G$  such that  $F_1 = \partial_{x_2} G$  and  $F_2 = -\partial_{x_1} G$ , and  $G \in L^2$  by Sobolev embedding because  $\nabla G = (-F_2, F_1) \in L^1$ .  $\square$

In fact in [1] and the subsequent work [2], [9], Bourgain-Brezis and van Schaftingen obtained some far-reaching generalizations of this simple lemma, and the latter is what we shall exploit in our more general result in this paper.

*Proof of Proposition 1.* Let  $f$  be as in the Proposition. Applying the lemma to  $f(t, \cdot)$  at each time  $t$ , we obtain a function  $g(t, \cdot)$  such that  $f_1 = \partial_{x_2} g$ ,  $f_2 = -\partial_{x_1} g$ , and  $\|g\|_{L_x^2} \leq C\|f\|_{L_x^1}$  at each time  $t$ . Now the classical energy estimate says that

$$\|u\|_{C_t^0 L_x^2} + \|\partial_t u\|_{C_t^0 \dot{H}_x^{-1}} \leq C \left( \|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-1}} + \|(-\Delta)^{-\frac{1}{2}} f\|_{L_t^1 L_x^2} \right).$$

Since for each fixed  $t$ ,

$$\|(-\Delta)^{-\frac{1}{2}} f\|_{L_x^2} = \|(-\Delta)^{-\frac{1}{2}} \nabla g\|_{L_x^2} \leq C\|g\|_{L_x^2} \leq \|\nabla f\|_{L_x^1}$$

by Sobolev embedding, our result follows.  $\square$

The key observation in proving Proposition 1 is that the coefficients of  $f$  are in  $\dot{H}^{-1}(\mathbb{R}^2)$  for all  $t$  under the given conditions. We remark that there are other situations under which the inhomogeneity of the wave equation lies in  $\dot{H}^{-1}(\mathbb{R}^2)$ ; one instance is given in the appendix.

In what follows, we derive improved Strichartz inequalities similar to Proposition 1, using generalizations of Lemma 1 by van Schaftingen.

## 1. STRICHARTZ ESTIMATES FOR THE WAVE EQUATION

In the sequel we shall consider vector fields  $f: \mathbb{R}^{1+n} \rightarrow \mathbb{R}^n$ . Our main result for the wave equation is the following:

**Theorem 1.** Suppose  $n \geq 2$ , and let  $u: \mathbb{R}^{1+n} \rightarrow \mathbb{R}^n$  be a (weak) solution of the system

$$\begin{cases} \square u = f \\ u|_{t=0} = u_0 \\ \partial_t u|_{t=0} = u_1 \end{cases}$$

where  $f(t, x): \mathbb{R}^{1+n} \rightarrow \mathbb{R}^n$  is a divergence free vector field for all  $t$ . Suppose  $s, k \in \mathbb{R}$ ,  $2 \leq q, \tilde{q} \leq \infty$ ,  $2 \leq r < \infty$ , and we assume further that  $\tilde{q} > \frac{4}{n-1}$  if  $n = 2$  or  $3$ . Suppose  $(q, r)$  satisfies the wave admissibility condition

$$\frac{1}{q} + \frac{n-1}{2r} \leq \frac{n-1}{4},$$

and the following scale invariance condition is verified:

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - s = \frac{1}{\tilde{q}'} + n - 2 - k.$$

Then

$$\begin{aligned} & \|u\|_{L_t^q L_x^r} + \|u\|_{C_t^0 \dot{H}_x^s} + \|\partial_t u\|_{C_t^0 \dot{H}_x^{s-1}} \\ & \leq C \left( \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}} + \|(-\Delta)^{\frac{k}{2}} f\|_{L_t^{\tilde{q}'} L_x^1} \right). \end{aligned}$$

To prove this, the starting point is the following result of van Schaftingen [9]:

**Theorem 2** (van Schaftingen). Let  $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a divergence free vector field with components in  $L^1$ . Then for any  $0 < \alpha < n$ ,

$$\|F\|_{\dot{W}^{-\alpha, \frac{n}{n-\alpha}}} \leq C \|F\|_{L^1}.$$

The Theorem was stated in [9] only for  $0 < \alpha \leq 1$ , but the rest of the theorem follows easily from Sobolev embedding of  $\dot{W}^{-\alpha, \frac{n}{n-\alpha}}$  into  $\dot{W}^{-\beta, \frac{n}{n-\beta}}$  in  $\mathbb{R}^n$  if  $0 < \alpha \leq \beta < n$ .

We also need the following version of the Strichartz estimate for the scalar equation. It is stated in Proposition 3.1 of Ginibre-Velo [6] for the non-endpoint case (where both  $(q, r), (\tilde{q}, \tilde{r}) \neq (2, \frac{2(n-1)}{n-3})$ ), and the endpoint case can be proved using the technology of Keel-Tao [7].

**Lemma 2.** Suppose  $n \geq 2$ , and let  $u: \mathbb{R}^{1+n} \rightarrow \mathbb{R}$  be a (weak) solution of

$$\begin{cases} \square u = h \\ u|_{t=0} = u_0 \\ \partial_t u|_{t=0} = u_1 \end{cases}$$

Suppose  $s, \gamma \in \mathbb{R}$ ,  $2 \leq q, \tilde{q} \leq \infty$ ,  $2 \leq r, \tilde{r} < \infty$ ,  $(q, r)$  and  $(\tilde{q}, \tilde{r})$  satisfy the wave admissibility conditions

$$\frac{1}{q} + \frac{n-1}{2r} \leq \frac{n-1}{4}, \quad \frac{1}{\tilde{q}} + \frac{n-1}{2\tilde{r}} \leq \frac{n-1}{4},$$

and the following scale invariance condition is verified:

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - s = \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - 2 - \gamma.$$

Then

$$\begin{aligned} & \|u\|_{L_t^q L_x^r} + \|u\|_{C_t^0 \dot{H}_x^s} + \|\partial_t u\|_{C_t^0 \dot{H}_x^{s-1}} \\ & \leq C \left( \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}} + \|(-\Delta)^{\frac{\gamma}{2}} h\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \right). \end{aligned}$$

For the convenience of the reader, we pause to outline a proof of the end-point case of Lemma 2:

*Proof of the end-point case of Lemma 2.* The desired estimate of  $\|u\|_{C_t^0 \dot{H}_x^s} + \|\partial_t u\|_{C_t^0 \dot{H}_x^{s-1}}$  follows from the statement of Corollary 1.3 of [7]. To prove

$$\|u\|_{L_t^q L_x^r} \leq C \|(-\Delta)^{\frac{\gamma}{2}} h\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'},}$$

one observes that since  $2 \leq q, \tilde{q} \leq \infty$ ,  $2 \leq r, \tilde{r} < \infty$ , one can restrict attention to the situation where the frequency support of  $h(t, \cdot)$  is contained in an annulus of size  $2^j$  by using the Littlewood-Paley square function. By scale invariance we can take  $j = 0$ . In that case  $(-\Delta)^{\frac{\gamma}{2}}$  on the right hand side can be dropped, and the result follows from Theorem 1.2 of [7].  $\square$

Theorem 1 can be seen as the limiting case of Lemma 2 when  $\tilde{r} = \infty$  except when  $(n, \tilde{q}, \tilde{r}) = (2, 4, \infty)$  or  $(3, 2, \infty)$ . It says one still has the Strichartz inequality if in addition  $f$  is a vector field at each time  $t$ , and  $f(t, x)$  is divergence free for all  $t$ .

*Proof of Theorem 1.* Assume  $n, q, \tilde{q}, r, k$  and  $s$  be as given in the statement of the Theorem. Then when  $n \geq 4$ , from  $2 \leq \tilde{q} \leq \infty$  one automatically has

$$\frac{n}{2} - \frac{2n}{(n-1)\tilde{q}} > 0,$$

and the same inequality holds when  $n = 2$  or  $3$  because then we assumed  $\tilde{q} > \frac{4}{n-1}$ . As a result, one can pick some  $\alpha \in (0, \frac{n}{2} - \frac{2n}{(n-1)\tilde{q}}]$ . Now let  $\tilde{r} = \frac{n}{\alpha}$ , and  $\gamma = k - \alpha$ . Then  $\tilde{r} < \infty$ ,  $\frac{1}{\tilde{q}} + \frac{n-1}{2\tilde{r}} \leq \frac{n-1}{4}$ , which in particular implies that  $\tilde{r} \geq 2$ . The scale invariance condition in Lemma 2 is also verified. Hence

$$\begin{aligned} & \|u\|_{L_t^q L_x^r} + \|u\|_{C_t^0 \dot{H}_x^s} + \|\partial_t u\|_{C_t^0 \dot{H}_x^{s-1}} \\ & \leq C \left( \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}} + \|(-\Delta)^{\frac{k-\alpha}{2}} f\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \right). \end{aligned}$$

Now invoking Theorem 2 and the divergence free condition on  $f$  for each time  $t$ , we get

$$\|(-\Delta)^{\frac{k-\alpha}{2}} f\|_{L_x^{\tilde{r}'}} \leq C \|(-\Delta)^{\frac{k}{2}} f\|_{L_x^1},$$

from which the desired inequality follows. Note this is possible because  $\alpha \in (0, n)$  automatically by our choice of  $\alpha$ .  $\square$

We remark that under the conditions of Theorem 1, we necessarily have  $s \geq 0$ , and when  $n \geq 3$  we necessarily have  $k > 0$ . In fact  $k \geq \frac{n-3}{2}$  when  $n \geq 3$ , and  $k = 0$  is impossible when  $n = 3$  because we assumed that  $\tilde{q} > 2$  when  $n = 3$ .

We also remark that in Theorem 1, when the initial conditions  $u_0$  and  $u_1$  are zero, one can actually obtain a wider range of exponents for which the desired inequality holds. This can be thought of as a limiting case of an inhomogeneous Strichartz estimate of Taggart [8], whose origin goes back to

the work of Foschi [5]. To illustrate this, we state the following Theorem in 3 space dimensions.

**Theorem 3.** *Suppose  $n = 3$ , and let  $u: \mathbb{R}^{1+3} \rightarrow \mathbb{R}^3$  be a (weak) solution of the system*

$$\begin{cases} \square u = f \\ u|_{t=0} = 0 \\ \partial_t u|_{t=0} = 0 \end{cases}$$

where  $f(t, x): \mathbb{R}^{1+3} \rightarrow \mathbb{R}^3$  is a divergence free vector field for all  $t$ . Suppose  $k \in \mathbb{R}$ ,  $1 < q, \tilde{q} \leq \infty$ ,  $2 \leq r < \infty$ , and

$$\frac{1}{q} + \frac{1}{\tilde{q}} < \min \left\{ 1, \frac{k+1}{2} \right\}.$$

Suppose further that  $(q, r)$  satisfies the wave acceptability condition

$$\frac{1}{q} + \frac{2}{r} < 1 \quad \text{or} \quad (q, r) = (\infty, 2),$$

and that the following scale invariance condition is verified:

$$\frac{1}{q} + \frac{3}{r} = 2 - k - \frac{1}{\tilde{q}}.$$

Then

$$\|u\|_{L_t^q L_x^r} \leq C \|(-\Delta)^{\frac{k}{2}} f\|_{L_t^{\tilde{q}'} L_x^1}.$$

To prove this, we need the following scalar inhomogeneous Strichartz estimate, which is a consequence of Corollary 8.7 of Taggart [8] in 3 space dimensions:

**Theorem 4** (Taggart). *Suppose  $n = 3$ , and let  $u: \mathbb{R}^{1+3} \rightarrow \mathbb{R}$  be a (weak) solution of*

$$\begin{cases} \square u = h \\ u|_{t=0} = 0 \\ \partial_t u|_{t=0} = 0. \end{cases}$$

Suppose  $\gamma \in \mathbb{R}$ ,  $1 < q, \tilde{q} \leq \infty$ ,  $2 \leq r, \tilde{r} < \infty$ ,

$$\frac{1}{q} + \frac{1}{\tilde{q}} < 1, \quad \text{and} \quad \frac{1}{q} + \frac{1}{\tilde{q}} \leq \frac{\gamma+1}{2}.$$

Suppose further that the exponents satisfy the wave acceptability condition

$$\frac{1}{q} + \frac{2}{r} < 1 \quad \text{or} \quad (q, r) = (\infty, 2),$$

$$\frac{1}{\tilde{q}} + \frac{2}{\tilde{r}} < 1 \quad \text{or} \quad (\tilde{q}, \tilde{r}) = (\infty, 2),$$

and that the following scale invariance condition is verified:

$$\frac{1}{q} + \frac{3}{r} = 2 - \gamma - \frac{1}{\tilde{q}} - \frac{3}{\tilde{r}}.$$

Then

$$\|u\|_{L_t^q L_x^r} \leq C \|(-\Delta)^{\frac{\gamma}{2}} h\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.$$

*Proof of Theorem 4.* Under the conditions of Theorem 4, one has

$$\frac{1}{q} + \frac{2}{r} < 1 \quad \text{or} \quad (q, r) = (\infty, 2),$$

$$\frac{1}{\tilde{q}} + \frac{2}{\tilde{r}} < 1 \quad \text{or} \quad (\tilde{q}, \tilde{r}) = (\infty, 2),$$

and

$$\frac{1}{r} + \frac{1}{\tilde{r}} \leq 1 - \frac{1}{q} - \frac{1}{\tilde{q}},$$

the last inequality following from the condition  $\frac{1}{q} + \frac{1}{\tilde{q}} \leq \frac{\gamma+1}{2}$  and the scale invariance condition. Thus one can find  $r_1 \leq r$ ,  $\tilde{r}_1 \leq \tilde{r}$  such that the wave acceptability conditions

$$\frac{1}{q} + \frac{2}{r_1} < 1 \quad \text{or} \quad (q, r_1) = (\infty, 2)$$

and

$$\frac{1}{\tilde{q}} + \frac{2}{\tilde{r}_1} < 1 \quad \text{or} \quad (\tilde{q}, \tilde{r}_1) = (\infty, 2),$$

are satisfied, with

$$\frac{1}{r_1} + \frac{1}{\tilde{r}_1} = 1 - \frac{1}{q} - \frac{1}{\tilde{q}}.$$

Clearly  $r_1, \tilde{r}_1 \in [2, \infty)$ . As a result, Corollary 8.7 of Taggart [8] applies, yielding Theorem 4.  $\square$

*Proof of Theorem 3.* Assume  $q, \tilde{q}, r$  and  $k$  be as given in the statement of the Theorem. Then since

$$\frac{1}{q} + \frac{1}{\tilde{q}} < \frac{k+1}{2} \quad \text{and} \quad \frac{1}{\tilde{q}} < 1,$$

one can pick a small  $\alpha > 0$  such that

$$\frac{1}{q} + \frac{1}{\tilde{q}} \leq \frac{(k-\alpha)+1}{2} \quad \text{and} \quad \frac{1}{\tilde{q}} + \frac{2\alpha}{3} < 1.$$

Now let  $\tilde{r} = \frac{3}{\alpha}$ , and  $\gamma = k - \alpha$ . Then  $\tilde{r} < \infty$ ,  $\frac{1}{q} + \frac{1}{\tilde{q}} \leq \frac{\gamma+1}{2}$ ,  $\frac{1}{q} + \frac{2}{\tilde{r}} < 1$ , which in particular implies that  $\tilde{r} > 2$ . The scale invariance condition in Theorem 4 is also verified. Hence

$$\|u\|_{L_t^q L_x^r} \leq C \|(-\Delta)^{\frac{k-\alpha}{2}} f\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.$$

Now invoking Theorem 2 and the divergence free condition on  $f$  for each time  $t$ , we get

$$\|(-\Delta)^{\frac{k-\alpha}{2}} f\|_{L_x^{\tilde{r}'}} \leq C \|(-\Delta)^{\frac{k}{2}} f\|_{L_x^1},$$

from which the desired inequality follows. Note this is possible because  $\alpha \in (0, 3)$  automatically by our choice of  $\alpha$ ; in fact  $\alpha < \frac{3}{2}$  since  $\frac{1}{q} + \frac{2\alpha}{3} < 1$ .  $\square$

## 2. STRICHARTZ ESTIMATES FOR THE SCHRÖDINGER EQUATION

Again, we consider vector fields  $f: \mathbb{R}^{1+n} \rightarrow \mathbb{R}^n$ . The main result is the following.

**Theorem 5.** Suppose  $n \geq 2$ , and  $u: \mathbb{R}^{1+n} \rightarrow \mathbb{R}^n$  is a (weak) solution of the system of Schrodinger equations

$$\begin{cases} i\partial_t u + \Delta u = f \\ u|_{t=0} = u_0, \end{cases}$$

where  $f(t, x): \mathbb{R}^{1+n} \rightarrow \mathbb{R}^n$  is a divergence free vector field for all  $t$ . Suppose  $2 \leq q, \tilde{q} \leq \infty$ ,  $2 \leq r < \infty$ ,  $s \geq 0$ ,  $k > s$ , and the following scale invariance conditions are satisfied:

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2} - s, \quad \frac{2}{\tilde{q}} = \frac{n}{2} - k + s.$$

Then

$$\|u\|_{C_t^0 \dot{H}_x^s} + \|u\|_{L_t^q L_x^r} \leq C \left( \|u_0\|_{\dot{H}^s} + \|(-\Delta)^{\frac{k}{2}} f\|_{L_t^{\tilde{q}'} L_x^1} \right).$$

In its proof we need Theorem 2 in the previous Section, as well as the following Strichartz inequality for the scalar Schrodinger equation (which follows from Corollary 1.4 of Keel-Tao [7] and the Sobolev inequality):

**Lemma 3.** Suppose  $n \geq 2$ , and  $u: \mathbb{R}^{1+n} \rightarrow \mathbb{R}$  is a (weak) solution of

$$\begin{cases} i\partial_t u + \Delta u = h \\ u|_{t=0} = u_0. \end{cases}$$

Suppose  $2 \leq q, \tilde{q} \leq \infty$ ,  $2 \leq r, \tilde{r} < \infty$ ,  $s \geq 0$ ,  $\gamma > s$ , and the following scale invariance conditions are satisfied:

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2} - s, \quad \frac{2}{\tilde{q}} + \frac{n}{\tilde{r}} = \frac{n}{2} - \gamma + s.$$

Then

$$\|u\|_{C_t^0 \dot{H}_x^s} + \|u\|_{L_t^q L_x^r} \leq C \left( \|u_0\|_{\dot{H}^s} + \|(-\Delta)^{\frac{\gamma}{2}} f\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \right).$$

Theorem 5 can be thought of as the limiting case of the above Lemma when  $\tilde{r} = \infty$ , which only works because we assumed that the inhomogeneity  $f(t, x)$  is a divergence free vector field at each time  $t$ .

*Proof of Theorem 5.* Assume  $n, q, \tilde{q}, r, k$  and  $s$  be as given in the statement of the Theorem. Then  $k - s > 0$ , one can pick some  $\alpha \in (0, \min\{k - s, \frac{n}{2}\}]$ . Now let  $\tilde{r} = \frac{n}{\alpha}$ , and  $\gamma = k - \alpha$ . Then  $2 \leq \tilde{r} < \infty$ ,  $\gamma > s$ , and

$$\frac{2}{\tilde{q}} + \frac{n}{\tilde{r}} = \frac{n}{2} - \gamma + s.$$

Hence

$$\|u\|_{C_t^0 \dot{H}_x^s} + \|u\|_{L_t^q L_x^r} \leq C \left( \|u_0\|_{\dot{H}^s} + \|(-\Delta)^{\frac{k-\alpha}{2}} f\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \right).$$

Now we invoke Theorem 2 and the divergence free condition on  $f$  at each time  $t$ ; this is possible because  $\alpha \in (0, n)$  automatically by our choice of  $\alpha$ . Thus we get

$$\|(-\Delta)^{\frac{k-\alpha}{2}} f\|_{L_x^{\tilde{r}'}} \leq C \|(-\Delta)^{\frac{k}{2}} f\|_{L_x^1},$$

from which the desired inequality follows.  $\square$

## 3. APPENDIX

In this appendix we prove another improved Strichartz inequality for the wave equation in  $\mathbb{R}^{1+2}$ . Here we only need to work with scalar equations.

**Proposition 2.** *Suppose  $u: \mathbb{R}^{1+2} \rightarrow \mathbb{R}$  satisfies*

$$\begin{cases} \square u = \det(\nabla_x F) \\ u|_{t=0} = u_0 \\ \partial_t u|_{t=0} = u_1 \end{cases}$$

where  $F$  is a map from  $\mathbb{R}^{1+2}$  to  $\mathbb{R}^2$ , and  $\det(\nabla_x F)$  denotes its Jacobian determinant in the  $x$  variable. Then

$$\|u\|_{C_t^0 L_x^2} + \|\partial_t u\|_{C_t^0 \dot{H}_x^{-1}} \leq C \left( \|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-1}} + \int \|\nabla_x F\|_{L_x^2}^2 dt \right).$$

It is clear that  $\|\nabla_x F\|_{L_x^2}^2$  controls the  $L_x^1$  norm of  $\det(\nabla_x F)$ , but unfortunately this is not enough if one wants to prove the Proposition. On the other hand, we claim

$$\|(-\Delta)^{-\frac{1}{2}} \det(\nabla_x F)\|_{L_x^2} \leq C \|\nabla_x F\|_{L_x^2}^2.$$

This follows from Wente's inequality; see e.g. Theorem 0.2 of Chanillo-Li [3]. Alternatively, since we are in 2 space dimensions, by compensation compactness (see Coifman-Lions-Meyer-Semmes [4]),  $\|\nabla_x F\|_{L_x^2}^2$  controls the Hardy  $\mathcal{H}_x^1$  norm of  $\det(\nabla_x F)$ , which in turn controls the negative Sobolev  $\dot{H}_x^{-1}$  norm of  $\det(\nabla_x F)$ , from which our claim follows. Arguing using the classical energy estimate as in the proof of Proposition 1, the desired estimate follows.

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